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A heuristic principle for the phase boundaries of pure and quenched bond-diluted spin models

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Abstract. Results from high-temperature series for a class of discrete, classical spin models on the square lattice suggest that the phase boundary is, at least in part, given by a simple heuristic principle. It is shown how this principle can be extended to triangular and hexagonal lattices by making use of duality and the star-triangle transformation. For the Potts model, the exact phase transition is recovered, not only for integer number of states, but also in the percolation limit. For more complicated models, a number of phase diagrams are derived, which are presumably exact except in regions where partially broken symmetry phases or massless phases intervene. The principle can also be applied to quenched bond-diluted spin models. The results for the Potts model are obtained for all three lattices and shown to be exact in certain limiting cases. The $M \to 1$ limit of the wreath product model $S(M) \wr S(M)$, which corresponds to 'double' percolation, is studied to provide an example for which the principle is completely incorrect.

1. Introduction

Discrete, classical spin models such as the Ashkin–Teller and clock models and their generalizations have complicated phase diagrams due to the possibility of the presence of partially broken symmetry and Kosterlitz–Thouless (massless) phases, see, e.g., [1–3] and references quoted there. Recently, the phase diagrams for a number of such models on the square lattice have been calculated approximately by means of 16-term high-temperature expansions for the free energy [4]. Although Kosterlitz–Thouless phases could not be identified with this method, the boundaries of the phases with partially broken symmetry as well as those where the symmetry of the spin model is broken completely could be detected. These latter phase boundaries always seem to coincide [4] with those given by a heuristic principle called pseudo-duality (at least for models which allow for a duality transformation). It is the purpose of the present paper to extend this principle to the hexagonal and triangular lattices and to apply it to a variety of pure and quenched bond-diluted spin models.

The organization of this paper is as follows: in section 2, the class of spin models that we are dealing with is briefly described together with the duality and star-triangle transformations for these models. The principle of pseudo-duality is defined for all three lattices. In section 3, this principle is applied to a variety of spin models. For the Potts model, the exact phase transition points are obtained for all lattices. Isotropic and anisotropic percolation on the three lattices can be considered as the limit (number of states) tends to 1 of the Potts model. In all cases, the exact percolation thresholds are recovered. A number of phase diagrams for models with two independent Boltzmann factors or coupling constants is proposed and the validity of the principle is discussed on the basis of the high-temperature series results of [4] and of the 'extra-special' points found in [5], where a star-triangle transformation is exact. In section 4, the principle is applied to the quenched bond-diluted Potts model. The derived phase boundaries are discussed and shown to be valid only in an approximate way, although they are exact at the percolation threshold and in the pure limit. In section 5, the $M \to 1$ limit of the wreath product model $S(M) \wr S(M)$ is studied and it is shown, that this corresponds to percolation on percolation clusters ('double' percolation), so that the phase boundaries are known exactly. In this case, the principle is shown to fail completely. In section 6, the results are discussed and the validity of the principle is elucidated.

2. Spin models, duality, the star-triangle transformation and pseudo-duality

Two classical spins, each of which can be in any of a finite number M of different states, have an interaction energy given by an $M \times M$ matrix $E(i, j)$. If a permutation g from the symmetric group $S(M)$ is such that the energy is invariant, i.e. such that

$$E(g(i), g(j)) = E(i, j) \qquad \text{for } i, j = 1, 2, \ldots, M \tag{2.1}$$

holds, then g is an element of the symmetry group G of the spin model. If G is transitive, it is of the type called permissible [3]. In this case, $E(i, j)$ is necessarily given in terms of the incidence matrices $M_k(i, j)$ of the graphs L_k with M vertices, which are obtained from a single edge (undirected) by application of the elements of G:

$$\begin{aligned} E(i, j) &= \sum_{k=1}^{s} E_k M_k(i, j) \\ M_k(i, j) &= \begin{cases} 1 & \text{if there is } g \in G \text{ with } (i, j) = (g(1), g(k)) \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \tag{2.2}$$

Here in the second equation $(i, j) = (j, i)$ denotes an undirected edge and s is the number of distinct graphs L_k obtained in this way. In writing the first of equations (2.1), use has also been made of the fact that $E(i, i) = 0$ can be chosen for all i by the transitivity of G.

In the present paper, we restrict ourselves to spin models for which the symmetry group G contains a regular (i.e. M element) Abelian subgroup A, which will be written additively. This ensures (see below) that the spin model has a dual with symmetry group $\tilde{G}$, which also contains A. The states of the spin model can then be identified with the elements of A and the $M_k(i, j)$ are necessarily linear combinations of the matrices $Q_a(i, j)$ defined as

$$Q_a(i, j) = \delta(j, i - a) \tag{2.3}$$

where the difference is to be interpreted in A. In fact, $Q_a(i, j) + Q_{-a}(i, j)$ is for $a \neq 0$ an incidence matrix for the group A. This implies that $E(i, j)$ depends only on the difference $i - j$ in A:

$$E(i, j) = E(i - j) = \sum_{a \neq 0} E_a Q_a(i, j) \qquad E_a = E_{-a}. \tag{2.4}$$

The Boltzmann factor corresponding to $E(i, j)$ is then

$$\Omega(i, j) = \exp -\beta E(i, j) = \sum_a \omega_a Q_a(i, j) \tag{2.5}$$

with

$$\omega_a = \begin{cases} 1 & \text{for } a = 0 \\ \exp -\beta E_a & \text{for } a \neq 0. \end{cases} \tag{2.6}$$

The matrix $\Omega(i, j)$ has normalized eigenvectors of the form

$$\mu_a(j) = M^{-1/2}\chi_a(j) \tag{2.7}$$

where the $\chi_a(j)$ are the characters of the (necessarily one-dimensional) irreducible representations of A. The eigenvalues of the Boltzmann factor matrix are

$$\lambda_a = \sum_b \omega_b \chi_a(-b). \tag{2.8}$$

These define a dual Boltzmann factor matrix by

$$\tilde{\Omega}(i, j) = \tilde{\Omega}(i - j) = \sum_a \tilde{\omega}_a Q_a(i, j), \tilde{\omega}_a = \lambda_a/\lambda_0. \tag{2.9}$$

The symmetry group $\tilde{G}$ of $\tilde{\Omega}(i, j)$ obviously contains A again. This duality transformation [6–8, 3] connects the partition function of the original spin model on a planar graph P with the partition function of the dual spin model on the dual graph $\tilde{P}$, which is obtained from P by putting a vertex in each face of P and connecting two such vertices by an edge of $\tilde{P}$ if the corresponding faces of P have an edge in common:

$$Z(P, \Omega) = M^{|V(P)|-|E(P)|-1}\lambda_0^{|E(P)|} Z(\tilde{P}, \tilde{\Omega}). \tag{2.10}$$

Here $V(P)$ and $E(P)$ are the vertex- and edge-sets of P, respectively. Defining

$$\gamma = \lim_{|V(P)|\to\infty} |V(P)|^{-1} \ln Z(P, \Omega) \tag{2.11}$$

equation (2.10) implies for the (self-dual) square lattice:

$$\gamma_{\text{sq}}(\Omega) = \ln(\lambda_0^2/M) + \gamma_{\text{sq}}(\tilde{\Omega}) \tag{2.12}$$

whereas the corresponding equation for the dual pair of the triangular and hexagonal lattices is

$$\gamma_{\text{hex}}(\Omega) = \tfrac{1}{2}\ln(\lambda_0^3/M) + \tfrac{1}{2}\gamma_{\text{tri}}(\tilde{\Omega}). \tag{2.13}$$

The corresponding results for the anisotropic lattices (for which there are different couplings depending on the orientation of the edges) are easily found as

$$\gamma_{\text{sq}}(\Omega_1, \Omega_2) = \ln(\lambda_0^{(1)}\lambda_0^{(2)}/M) + \gamma_{\text{sq}}(\tilde{\Omega}_2, \tilde{\Omega}_1) \tag{2.14}$$

and

$$\gamma_{\text{hex}}(\Omega_1, \Omega_2, \Omega_3) = \tfrac{1}{2}\ln(\lambda_0^{(1)}\lambda_0^{(2)}\lambda_0^{(3)}/M) + \tfrac{1}{2}\gamma_{\text{tri}}(\tilde{\Omega}_3, \tilde{\Omega}_2, \tilde{\Omega}_1). \tag{2.15}$$

For the square lattice, the pseudo-duality condition found in [4] is simply the vanishing of the extra term on the right-hand-side of equations (2.12) or (2.14):

$$\ln(\lambda_0^{(1)}\lambda_0^{(2)}/M) = 0 \qquad \text{or} \qquad \lambda_0^{(1)}\lambda_0^{(2)} = M \text{ (pseudo-duality for sq).} \tag{2.16}$$

It is, for a self-dual model, a necessary, but by no means sufficient, condition for the self-duality relation $\Omega_1 = \tilde{\Omega}_2$ (which implies $\Omega_2 = \tilde{\Omega}_1$) to be valid.

In order to extend the notion of pseudo-duality to the triangular and hexagonal lattices, another ingredient is needed. This is provided by the star-triangle relation obtained by summing over all states of the central spin in a star configuration [5]:

$$\sum_\alpha \Omega_1(i,\alpha)\Omega_2(j,\alpha)\Omega_3(k,\alpha) = C\Omega_1^*(i-j)\Omega_2^*(j-k)\Omega_3^*(k-i)W(i,j,k). \tag{2.17}$$

Here the Ω_i^* are effective pair interactions on a triangle and $W(i,j,k)$ is the remaining three-spin interaction. The requirements $\Omega_i^*(0) = 1$ and $W(i,i,i) = 1$ fix the constant C as

$$C = \sum_a \omega_a^{(1)}\omega_a^{(2)}\omega_a^{(3)}.$$

Equation (2.17) implies

$$\gamma_{\text{hex}}(\Omega_1,\Omega_2,\Omega_3) = \tfrac{1}{2}\ln C + \tfrac{1}{2}\gamma_{\text{tri}}(\Omega_3^*,\Omega_2^*,\Omega_1^*). \tag{2.18}$$

Comparing this with the duality relation of (2.15) suggests as pseudo-duality relation for the hexagonal lattice:

$$\ln(\lambda_0^{(1)}\lambda_0^{(2)}\lambda_0^{(3)}/M) = \ln\left(\sum_a \omega_a^{(1)}\omega_a^{(2)}\omega_a^{(3)}\right) \qquad \text{(pseudo-duality for hex).} \tag{2.19}$$

From this, the duality transformation of (2.9) gives, for the triangular lattice, the relation

$$\ln\left(\sum_a \lambda_a^{(1)}\lambda_a^{(2)}\lambda_a^{(3)}\right) = 2\ln(M) \qquad \text{(pseudo-duality for tri).} \tag{2.20}$$

For a self-dual model, equations (2.19) or (2.20) are necessary but not sufficient for the existence of a star-triangle transformation, i.e. for the vanishing of the three-spin interaction, $W(i,j,k) = 1$ for all i, j and k [5].

The heuristic nature of the above 'derivations' of the pseudo-duality relations of equations (2.16), (2.19), (2.20) is obvious. It is, therefore, not possible to comment on the physical content of these relations in an *a priori* fashion. The rest of this paper is concerned with the exploration of the consequences of these relations and, hence, to obtain some insight into the range of validity of them as indicators of phase boundaries.

3. Pseudo-duality for some spin models

3.1. The Potts model

The Potts model has the full symmetric group $S(M)$ as symmetry group, so that its Boltzmann factor matrix has the form

$$\Omega(i,j) = \begin{cases} 1 & \text{for } i = j \\ \omega & \text{for } i \neq j. \end{cases} \tag{3.1}$$

Since $S(M)$ contains every regular Abelian group (for instance, the cyclic group $Z(M)$), the model is self-dual with dual Boltzmann factor matrix

$$\tilde{\Omega}(i,j) = \begin{cases} 1 & \text{for } i \neq j \\ \tilde{\omega} = (1-\omega)/[1+(M-1)\omega] \end{cases} \tag{3.2}$$

since the eigenvalues of the matrix of equation (3.1) are

$$\lambda_0 = 1 + (M-1)\omega \qquad \lambda_a = 1 - \omega \qquad \text{for } a \neq 0. \tag{3.3}$$

The pseudo-duality conditions can, after some algebra, be cast in the forms

$$(M-1)\omega^{(1)}\omega^{(2)} + \omega^{(1)} + \omega^{(2)} = 1 \qquad \text{(sq)} \tag{3.4}$$

$$(M^2 - 3M + 1)\omega^{(1)}\omega^{(2)}\omega^{(3)} + (M-1)(\omega^{(1)}\omega^{(2)} + \omega^{(1)}\omega^{(3)} + \omega^{(2)}\omega^{(3)}) + \omega^{(1)} + \omega^{(2)} + \omega^{(3)} = 1 \qquad \text{(hex)} \tag{3.5}$$

$$(M-2)\omega^{(1)}\omega^{(2)}\omega^{(3)} + \omega^{(1)}\omega^{(2)} + \omega^{(1)}\omega^{(3)} + \omega^{(2)}\omega^{(3)} = 1 \qquad \text{(tri)}. \tag{3.6}$$

Equation (3.4) just defines the self-dual Potts model, whereas equations (3.5) or (3.6) ensure the existence of a star-triangle transformation [5]. In all cases, equations (3.4)–(3.6) are known (or strongly conjectured) to define the exact phase transition points for the Potts model on these lattices [9–12].

The percolation thresholds follow from equations (3.4)–(3.6) by setting the probability for a bond of orientation i, p_i, equal to $1-\omega^{(i)}$ and taking the limit $M \to 1$; the results are

$$p_1 + p_2 = 1 \qquad \text{(sq)} \tag{3.7}$$

$$-p_1p_2p_3 + p_1p_2 + p_1p_3 + p_2p_3 = 1 \qquad \text{(hex)} \tag{3.8}$$

$$-p_1p_2p_3 + p_1 + p_2 + p_3 = 1 \qquad \text{(tri)}. \tag{3.9}$$

For the isotropic case, this gives the exact percolation thresholds [13]:

$$p_c = \tfrac{1}{2}\ \text{(sq)} \qquad p_c = 1 - 2\sin(\pi/18)\ \text{(hex)} \qquad p_c = 2\sin(\pi/18)\ \text{(tri)}. \tag{3.10}$$

The results of this subsection are also special cases of those obtained for the Potts model on the chequerboard lattice in [14] and of those obtained for a Potts model with additional three-spin interactions in half of the triangles of the triangular lattice in [15] and [16].

3.2. Wreath product models

In this subsection the simplest wreath product models of the type $S(M_1)\wr S(M_2)$, which were first described by Zamolodchikov and Monastyrskii [17], are considered. For these models, there are two independent coupling constants or Boltzmann factors ω_1 and ω_2, which are each associated with a graph as in equation (2.2), where $s = 2$ now. The graph L_1 consists of M_2 copies of the complete graph on M_1 vertices, whereas L_2 is the complement of L_1 in the complete graph on M_1M_2 vertices. Explicitly, the Boltzmann factor matrix can be written:

$$\Omega(i,j) = \begin{cases} 1 & \text{for } i = j \\ \omega_1 & \text{for } i \equiv j \bmod M_2 \text{ and } i \neq j \\ \omega_2 & \text{otherwise.} \end{cases} \tag{3.11}$$

Since $S(M_1) \wr S(M_2)$ contains the regular Abelian subgroup $Z(M_1) \otimes Z(M_2)$, a duality transformation exists; the eigenvalues of the Boltzmann factor matrix are:

$$
\begin{aligned}
&\lambda_0 = 1 + (M_1 - 1)\omega_1 + M_1(M_2 - 1)\omega_2 \qquad \text{(non-degenerate)} \\
&\lambda_1 = 1 + (M_1 - 1)\omega_1 - M_1\omega_2 \qquad ((M_2 - 1)\text{-fold degenerate}) \\
&\lambda_2 = 1 - \omega_1 \qquad (M_2(M_1 - 1)\text{-fold degenerate}).
\end{aligned}
\tag{3.12}
$$

Equation (2.9) now shows that the symmetry group of the dual model is $S(M_2) \wr S(M_1)$, so that the model is self-dual only for $M_1 = M_2$.

For the isotropic square lattice, the pseudo-duality relation of equation (2.16) gives the straight line

$$
(M_1 - 1)\omega_1 + M_1(M_2 - 1)\omega_2 = (M_1 M_2)^{1/2} - 1. \tag{3.13}
$$

For the self-dual case ($M_1 = M_2$), this result can already be found in [17]; it is then the self-dual line. For the isotropic hexagonal and triangular lattices, equations (2.19) and (2.20) yield rather involved expressions, which will not be given here explicitly for the general case. The resulting curves and straight lines are shown in figures 1–4 for the cases $(M_1, M_2) = (2, 2), (2, 3), (3, 2)$ and (3,3), respectively. Also shown are the Potts model points on the line $\omega_1 = \omega_2$ as well as the special points, where the model reduces to a simpler one or factorizes into a product of simpler ones. For the case $M_1 = M_2 = 2$ in figure 1, which is the symmetric Ashkin–Teller model, there are three such special points for each lattice. These are the points I on $\omega_2 = 0$ and its dual $\tilde{I}$ on $\omega_1 = 1$, where the model reduces to the Ising model, and I_1 (self-dual), where a factorization into two (critical) Ising models occurs; I_1 always lies on the curve $\omega_1 = \omega_2^2$. For this model only, the pseudo-dual curves for the hexagonal and triangular lattices can be given explicitly:

$$
\omega_1 = 1 + \omega_2 - (4\omega_2 + 5\omega_2^2)^{1/2} \text{ (hex)} \qquad \omega_1 = 1 - 2\omega_2^2 \text{ (tri)}. \tag{3.14}
$$

The phase boundaries for the dual pair $S(2) \wr S(3)$, $S(3) \wr S(2)$ are shown as figures 2 and 3; these models can reduce to (critical) Ising and three-state Potts models. These special points are I on $\omega_2 = 0$ and $\tilde{P}(3)$ on $\omega_1 = 1$ for $S(2) \wr S(3)$ and their duals $\tilde{I}$ on $\omega_1 = 1$ and $P(3)$ on $\omega_2 = 0$ for $S(3) \wr S(2)$. In the case of the self-dual model $S(3) \wr S(3)$, there is a dual pair of special points, $P(3)$ on $\omega_2 = 0$ and $\tilde{P}(3)$ on $\omega_1 = 1$, where the model reduces to a critical three-state Potts model, see figure 4.

In the case of the square lattice, the high-temperature series analysis [4] indicates that the portion of the pseudo-dual line is indeed a phase boundary for $\omega_2 \geqslant \omega_1$ with complete symmetry breaking. In the region $\omega_1 > \omega_2$, there are two phase boundaries, starting at the Potts model point $P(M_1 M_2)$ and ending at the appropriate points on $\omega_1 = 1$ and $\omega_2 = 0$, which enclose a phase with partially broken symmetry. Figures 1–4 suggest that the same is true for the triangular and hexagonal lattices, so that the curves indicated are, for $\omega_2 \geqslant \omega_1$, presumably the exact phase transition curves for complete symmetry breaking.

For a self-dual model, there may be points in the phase diagram, where a star-triangle transformation is exact [5]. If these points are not Potts model or trivial extreme points, they are called 'extra-special'. In figure 4 the extra-special points for the $S(3) \wr S(3)$ model have also been indicated. It has been conjectured that these are the exact points where a Kosterlitz–Thouless or massless phase (characterized by algebraic decay of the correlation functions) first appears. Such massless phases may be expected for models which contain a regular Abelian subgroup $Z(M)$ with $M \geqslant 5$ [2,5]. Since the $S(3) \wr S(3)$ model contains $Z(9)$, a massless phase may be expected here. If the point T really is the apex of a massless phase, then this phase seems to be embedded completely inside the partially broken symmetry phase.

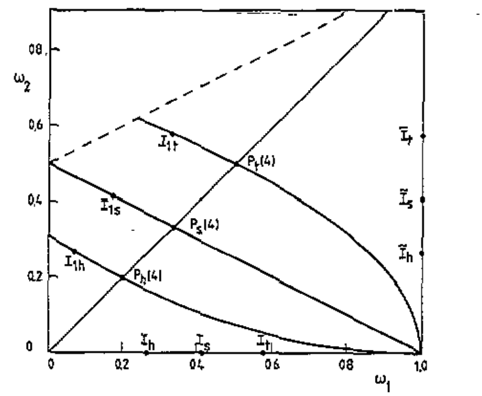

Figure 1. Phase diagrams obtained from pseudo-duality for the square (s), hexagonal (h) and triangular (t) lattices for the symmetric Ashkin–Teller model with symmetry group $S(2) \wr S(2)$. Also shown are the four-state Potts phase transition points as well as the points I and $\tilde{I}$, which are Ising critical points and I_1, where the model factorizes into two critical Ising models.

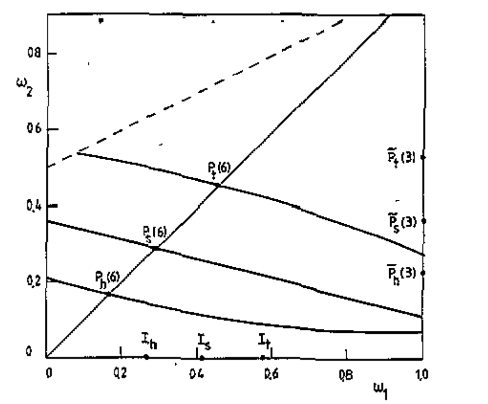

Figure 2. As for figure 1 for the $S(2) \wr S(3)$ model. Special points are the six-state Potts model transition point and the points I and $\tilde{P}(3)$, where the model reduces to a critical Ising model and to a critical three-state Potts model, respectively.

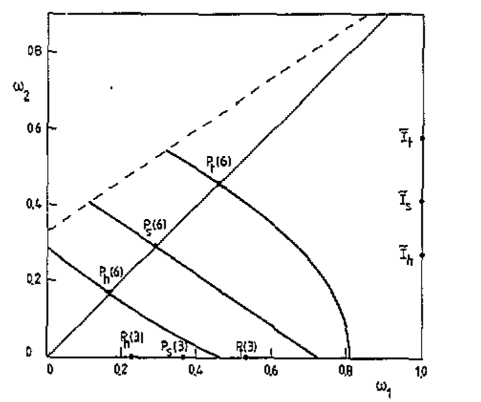

Figure 3. *As for figure 1 for the $S(3) \wr S(2)$ model.* Special points are the duals of those in figure 2.

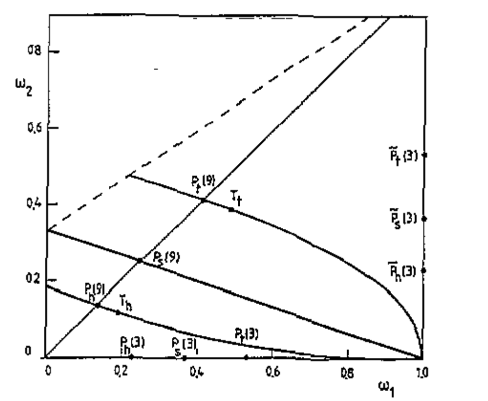

Figure 4. *As for figure 1 for the self-dual $S(3) \wr S(3)$* model. Special points indicated are the nine-state Potts model transition point $P(9)$, the points where the model reduces to a critical three-state Potts model ($P(3)$ and $\tilde{P}(3)$) as well as the points T_h and T_t, which are extra-special, i.e. where a star-triangle transformation exists.

3.3. Models with primitive symmetry groups

For models with a primitive symmetry group, no partially broken symmetry phases are possible [3]. In this subsection, we consider the two models with two independent coupling constants, which contain regular Abelian subgroups and have not more than 10 states. These are the five-state clock model with symmetry group $Z(5)$ and a nine-state model [3], both of which are self-dual. For the $Z(5)$ model, the Boltzmann factor matrix has the form

$$\Omega)i,j) = \begin{cases} 1 & \text{for } i = j \\ \omega_1 & \text{for } i - j \equiv \pm 1 \bmod 5 \\ \omega_2 & \text{for } i - j \equiv \pm 2 \bmod 5. \end{cases} \tag{3.15}$$

This matrix has the eigenvalues

$$\begin{aligned} \lambda_0 &= 1 + 2\omega_1 + 2\omega_2 && \text{(non-degenerate)} \\ \lambda_1 &= 1 + \tfrac{1}{2}(\sqrt{5} - 1)\omega_1 - \tfrac{1}{2}(\sqrt{5} + 1)\omega_2 && \text{(twofold degenerate)} \\ \lambda_2 &= 1 - \tfrac{1}{2}(\sqrt{5} + 1)\omega_1 + \tfrac{1}{2}(\sqrt{5} - 1)\omega_2 && \text{(twofold degenerate).} \end{aligned} \tag{3.16}$$

For the isotropic square lattice, the pseudo-duality relation of equation (2.16) is the self-duality condition again:

$$\omega_1 + \omega_2 = \tfrac{1}{2}(\sqrt{5} - 1). \tag{3.17}$$

For the isotropic triangular and hexagonal lattices, equations (2.19), (2.20) and (3.15), (3.16) give the curves shown (together with the line of equation (3.17)) in figure 5. Also shown are the critical five-state Potts model point at $P(5)$ and the extra-special points T_1 and T_2 found in [5]. For the square lattice, the short series [4] picked up the phase transition line of equation (3.17) only in the neighbourhood of $P_s(5)$. It does not seem unreasonable to suppose that the curves in figure 5 really represent the phase transition between the two extra-special points T_1 and T_2, including the Potts model point $P(5)$. These extra-special points could then indeed be the apexes of the massless phases for the present model [18].

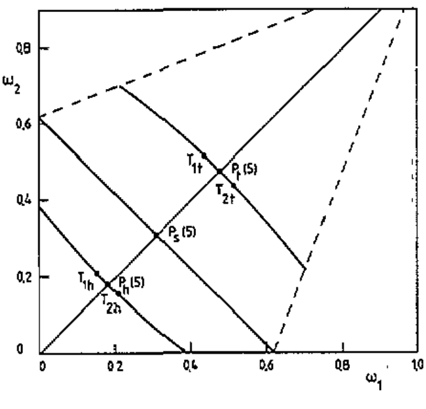

Figure 5. As for figure 1 for the $Z(5)$ model. The special points are $P(5)$ for the critical five-state Potts model and the extra-special points T_{1h}, T_{2h}, T_{1t} and T_{2t}.

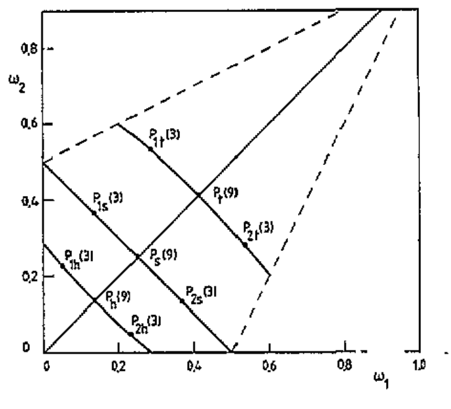

Figure 6. As for figure 1 for the primitive nine-state model. Here $P(9)$ is the nine-state Potts transition point, whereas the model factorizes into two critical three-state Potts models in $P_1(3)$ and $P_2(3)$.

The nine-state primitive model alluded to above [3] is defined here by giving its Boltzmann factor matrix explicitly:

$$\Omega(i,j) = \begin{cases} 1 & \text{for } i = j \\ \omega_1 & \text{for } (i,j) \text{ an edge of one of the triangles} \\ & \quad (123), (456), (789), (147), (258) \text{ or } (369) \\ \omega_2 & \text{otherwise.} \end{cases} \tag{3.18}$$

The symmetry group of this model (which is technically the automorphism group of the covering graph of the complement of two triangles) contains $Z(3) \otimes Z(3)$, but not $Z(9)$, so that no Kosterlitz–Thouless phases are to be expected. The eigenvalues of the matrix of equation (3.18) are

$$\begin{aligned} \lambda_0 &= 1 + 4\omega_1 + 4\omega_2 && \text{(non-degenerate)} \\ \lambda_1 &= 1 + \omega_1 - 2\omega_2 && \text{(fourfold degenerate)} \\ \lambda_2 &= 1 - 2\omega_1 + \omega_2 && \text{(fourfold degenerate).} \end{aligned} \tag{3.19}$$

The pseudo-dual curves derived from equations (2.15), (2.19), (2.20) and (3.18), (3.19) are shown in figure 6. Also indicated are the nine-state Potts critical point, $P(9)$, and the points where the model factorizes into a product of critical three-state Potts models ($P_1(3)$ and $P_2(3)$ on the curves $\omega_1 = \omega_2^2$ and $\omega_2 = \omega_1^2$, respectively). For the present case, figure 6 may be expected to be the exact phase diagram, since there are no partially broken symmetry or massless phases for this model.

4. Quenched bond-diluted spin models

It is tempting to extend the pseudo-duality relations to cases with quenched bond disorder. For the square lattice, equation (2.16) would be replaced by

$$\langle \ln \lambda_0^{(1)} \rangle + \langle \ln \lambda_0^{(2)} \rangle = \ln M \qquad \text{(sq)} \tag{4.1}$$

where the symbol $\langle\,\rangle$ denotes an average over bond disorder. Similarly, for the hexagonal lattice, equation (2.19) would be replaced by

$$\sum_{i=1}^{3} \langle \ln \lambda_0^{(i)} \rangle - \left\langle \ln \sum_a \omega_a^{(1)} \omega_a^{(2)} \omega_a^{(3)} \right\rangle = \ln M \qquad \text{(hex)} \tag{4.2}$$

whereas equation (2.20) for the triangular lattice becomes

$$\left\langle \ln \sum_a \lambda_a^{(1)} \lambda_a^{(2)} \lambda_a^{(3)} \right\rangle = 2 \ln M \qquad \text{(tri).} \tag{4.3}$$

In the following, we will consider equations (4.1)–(4.3) only for the Potts model and only for the bond-diluted case, so that a bond is present with probability p (or p_i in the anisotropic case) and absent (this means $\omega = 1$ or $\omega^{(i)} = 1$) with probability $1 - p$ or $1 - p_i$. In the most general anisotropic case, (4.1)–(4.3) yield the results:

$$p_1 \ln[1 + (M-1)\omega^{(1)}] + p_2 \ln[1 + (M-1)\omega^{(2)}] = (p_1 + p_2 - 1) \ln M \tag{4.4}$$

$$\begin{aligned} &- p_1 p_2 p_3 \ln[1 + (M-1)\omega^{(1)}\omega^{(2)}\omega^{(3)}] - \sum_{k=1}^{3} p_i p_j (1 - p_k) \ln[1 + (M-1)\omega^{(i)}\omega^{(j)}] \\ &\qquad + \sum_{k=1}^{3} p_k (p_i + p_j - p_i p_j) \ln[1 + (M-1)\omega^{(k)}] \\ &\qquad = (-p_1 p_2 p_3 + p_1 p_2 + p_1 p_3 + p_2 p_3 - 1) \ln M \end{aligned} \tag{4.5}$$

$$\begin{aligned} &p_1 p_2 p_3 \ln[1 + (M-1)(\omega^{(1)}\omega^{(2)} + \omega^{(1)}\omega^{(3)} + \omega^{(2)}\omega^{(3)}) + (M-1)(M-2)\omega^{(1)}\omega^{(2)}\omega^{(3)}] \\ &\qquad + \sum_{k=1}^{3} p_k (1 - p_i p_j) \ln[1 + (M-1)\omega^{(k)}] \\ &\qquad = (-p_1 p_2 p_3 + p_1 + p_2 + p_3 - 1) \ln M. \end{aligned} \tag{4.6}$$

In equations (4.5) and (4.6), i and j are those indices from (123) not equal to k. Equations (4.4)–(4.6) are consistent with the percolation results of equations (3.7)–(3.9) in the sense that no solution for the Boltzmann factors $\omega^{(i)}$ is possible if the probabilities p_i are such that no percolation occurs; exactly at the percolation thresholds, all $\omega^{(i)}$ are necessarily equal to zero.

For the isotropic case ($p_1 = p_2 = p, \omega^{(1)} = \omega^{(2)} = \omega$), equation (4.4) has the explicit solution

$$\omega = (M^{(2p-1)/(2p)} - 1)/(M-1) \qquad p \geqslant \tfrac{1}{2} \qquad (\mathrm{sq}) \tag{4.7}$$

This is identical to a heuristic result due to Southern [19] derived by means of the replica trick. For the Ising model, $M = 2$, this result was first derived by Nishimori [20], who also derived the general equation (4.1) for the isotropic Ising model [21]. The corresponding equations determining ω in the cases of the other two lattices are (with $p_1 = p_2 = p_3 = p$ and $\omega^{(1)} = \omega^{(2)} = \omega^{(3)}\omega$):

$$\begin{aligned} - p^3 \ln[1 + (M-1)\omega^3] - 3p^2(1-p)\ln[1+(M-1)\omega^2] + 3p^2(2-p)\ln[1+(M-1)\omega] \\ = (-p^3 + 3p^2 - 1)\ln(M) \qquad (\mathrm{hex}) \end{aligned} \tag{4.8}$$

$$\begin{aligned} p^3 \ln[1 + 3(M-1)\omega^2 + (M-1)(M-2)\omega^3] + 3p(1-p^2)\ln[1+(M-1)\omega] \\ = (-p^3 + 3p - 1)\ln(M) \qquad (\mathrm{tri}). \end{aligned} \tag{4.9}$$

As shown in figure 7 for the Ising ($M = 2$) and four-state Potts models, the solutions for ω of equations (4.8) and (4.9) smoothly interpolate between $\omega = 0$ at the percolation threshold and the exact critical ω_c at $p = 1$ in a way similar to the one given by equation (4.7).

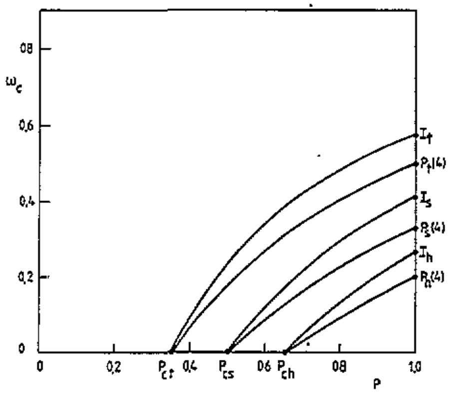

Figure 7. The phase boundaries for the quenched bond-diluted Potts model as obtained from pseudo-duality for the three planar lattices (h,s,t) for $M = 2$ (Ising model) and $M = 4$. p_c is the percolation threshold, I and $P(4)$ are the pure model transition points.

In the neighbourhood of the percolation threshold p_c, equations (4.7)–(4.9) all imply the following behaviour for ω:

$$\omega = (p - p_c)(\ln(M))/[p_c(M-1)] + 0((p-p_c)^2) \tag{4.10}$$

which is in accordance with renormalization group [22] and replica trick [23] arguments. In the neighbourhood of $p = 1$, equation (4.7) is known to be incorrect for $M = 2, 3$ and 4 [23]. For $M = 2$, equation (4.8) can also be compared with an exact result in this limit. Here also a (small) discrepancy in $T_c^{-1}(\partial T_c/\partial p)$ at $p = 1$ is found: the exact value is 1.579

[24], whereas (4.8) gives a value of 1.600. A third limit, $M \to 1$, describes the doubly diluted case; equations (4.7)–(4.9) all reduce to the exact result [23, 25]

$$p(1-\omega) = p_c \tag{4.11}$$

in this limit.

It is concluded that equations (4.8) and (4.9) are useful approximate relations, which are, just as equation (4.7) is exact in a number of limiting cases ($p - p_c$ small, $p = 1$ and $M \to 1$). This can then also be concluded for the general anisotropic results of equations (4.4)–(4.6). We will return to the validity of the pseudo-duality relations (including the Nishimori and Southern expressions) in the quenched bond-diluted case in section 6.

5. A counterexample: double percolation

In this section, we study the $M \to 1$ limit of the self-dual $S(M) \wr S(M)$ wreath product spin model introduced in section 3.2. To this end, the Boltzmann factor matrix of (3.11) is rewritten as ($M_1 = M_2 = M$):

$$\Omega(i, j) = \omega_2 + (1-\omega_1)\delta(i, j) + (\omega_1 - \omega_2)\delta_M(i, j) \tag{5.1}$$

where $\delta(i, j)$ is the Kronecker delta and $\delta_M(i, j)$ is given by

$$\delta_M(i, j) = \begin{cases} 1 & \text{if } i \equiv j \bmod M \\ 0 & \text{otherwise.} \end{cases} \tag{5.2}$$

The partition function for this model on a graph or lattice L is then

$$Z(L) = \sum_{\substack{i_v = 1, \ldots, M^2 \\ v \in V}} \prod_{e \in E} [\omega_2 + (1-\omega_1)\delta(i_{v_1}, i_{v_2}) + (\omega_1 - \omega_2)\delta_M(i_{v_1}, i_{v_2})] \tag{5.3}$$

Here V and E are the vertex- and edge-sets of L, respectively, and v_1 and v_2 are the vertices at the ends of the edge e. Equation (5.3) can be rewritten using edge variables μ_e, which take the values 0 and 1:

$$Z(L) = \sum_{i_v, v \in V} \sum_{\mu_e = 0,1} \prod_{e \in E} \omega_2^{1-\mu_e} [(1-\omega_1)\delta(i, j) + (\omega_1 - \omega_2)\delta_M(i, j)]^{\mu_e}. \tag{5.4}$$

Now the edges present (i.e. those with $\mu_e = 1$) will define a number of connected components. This number will be denoted by $N_c\{\mu_e\}$. The form in the square brackets in (5.4) is such that if a summation over the spins is done for each connected component, then the result will be M times that obtained by summing a product of factors of the form

$$A(i, j) = (1-\omega_1)\delta(i, j) + (\omega_1 - \omega_2) \tag{5.5}$$

but with the vertex variables in the range $1, \ldots, M$. Equation (5.5) is just the Boltzmann factor matrix for an M-state Potts model up to a factor:

$$A(i, j) = (1-\omega_2) \begin{cases} 1 & \text{for } i \neq j \\ \omega' = (\omega_1 - \omega_2)/(1-\omega_2). & \end{cases} \tag{5.6}$$

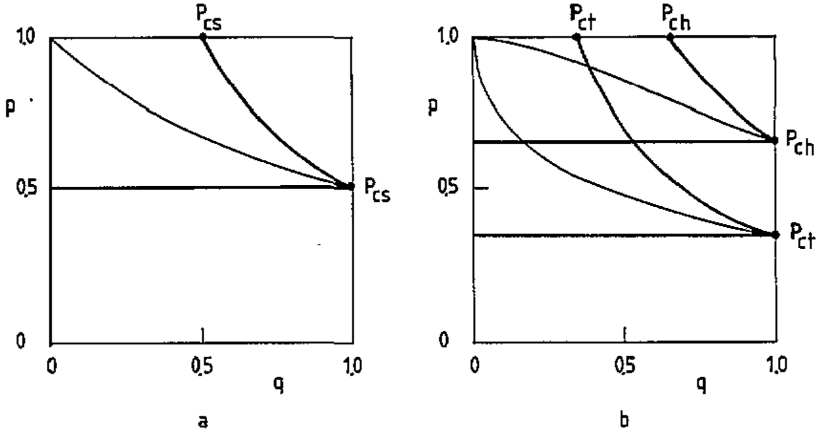

Figure 8. Phase diagrams for double percolation on the planar lattices (h,s,t). The bold curves are exact, the fine ones follow from pseudo-duality. The percolation thresholds are at p_c.

Therefore, equation (5.4) is equivalent to

$$Z(L) = \sum_{\mu_e=0,1} \omega_2^{\sum_e(1-\mu_e)}(1-\omega_2)^{\sum_e \mu_e} M^{N_c(\mu_e)} \prod_{k=1}^{N_c} Z_{\text{Potts}}^{(k)}(\omega') \tag{5.7}$$

with $Z_{\text{Potts}}^{(k)}(\omega')$ the partition function for the M-state Potts model on the kth connected component. It is noted here that ω' has a physically meaningful value only for $\omega_1 \geqslant \omega_2$, i.e. in the region where the $S(M) \wr S(M)$ model has an intervening phase with partially broken symmetry, see section 3.2.

In the limit $M \to 1$, (5.7) describes 'double' percolation: the first factors amount to a probability $p = 1 - \omega_2$ for the presence of a bond and the factor $M^{N_c(\mu_e)}$ will provide a phase transition at $p = p_c$. If $\omega' > 0$ holds, the product of Potts model partition functions can be interpreted as a secondary dilution on percolation clusters; those bonds, which are left over on the first dilution, will now have a probability of being still present reduced by a factor

$$q = 1 - \omega' = (1-\omega_1)/(1-\omega_2) = (1-\omega_1)/p. \tag{5.8}$$

There will then be a second phase transition at $pq = p_c$, i.e. for $pq \geqslant p_c$ there is still an infinite percolation cluster. Figure 8 shows these phase transition boundaries for the planar lattices as bold curves. Also shown here are the (fine) curves given by the pseudo-duality relations of equations (2.15)–(2.19), (2.20), which are given by

$$\begin{aligned} &p = 1/(1+q) \qquad \text{(sq)} \\ &p^3(1+q^3) - 3p^2(1+q^2) + 2 = 0 \qquad \text{(hex)} \\ &p^3(1+q^3) + 3p(1+q) + 2 = 0 \qquad \text{(tri)} \end{aligned} \tag{5.9}$$

for the present case. The failure of the pseudo-duality relations is complete here for $q < 1$; this is obviously due to the fact that a percolation interpretation of q (equation (5.8)) is possible only for $\omega_1 \geqslant \omega_2$, so that, for integer M, one always stays in a region of the phase diagram with intervening partially broken symmetry phase. This then seems to carry through in the limit $M \to 1$.

6. Discussion

The successes and failures can be understood qualitatively in a renormalization-group context, if one supposes that renormalization keeps a part of the subspace defined by the pseudo-duality relations invariant. This part is not attracted to a non-trivial fixed point corresponding to a phase, which is either massless or has partially broken symmetry. This would explain the failure of the principle in case such phases are present.

The situation for the quenched bond-diluted systems studied in section 4 is not yet clear on this basis. Here the replica method can be used to elucidate this problem: the original problem is the calculation of the average of ln Z over the disorder. The replica trick replaces this average by

$$\left\langle \lim_{n\to 0}(Z^n - 1)/n\right\rangle. \tag{6.1}$$

Now for n integer, the average of Z^n is, in the bond-diluted case,

$$\langle Z^n\rangle = \sum_{\substack{i_v^1,\ldots,i_v^n\\ v\in V}} \prod_{e\in E}\left[p\prod_{k=1}^{n}\Omega\left(i_{v_1}^k, i_{v_2}^k\right) + 1 - p\right]. \tag{6.2}$$

Here the notation is as in (5.3). This is simply the partition function of a spin model with M^n states per spin and with the Boltzmann factor matrix given by

$$\Omega_n(i^1,\ldots,i^n; j^1,\ldots,j^n) = p\prod_{k=1}^{n}\Omega(i^k, j^k) + 1 - p. \tag{6.3}$$

It is now not difficult to see that the equations derived in section 4 would be correct if the pseudo-duality relations are valid for all models of the type of equation (6.3) for all n (and also, formally, for $n \to 0$). The symmetry group of the model in equation (6.3) is the n-fold wreath product of the symmetry group G of $\Omega(i, j)$ with itself. For these models, phases with partially broken symmetry do occur (and also massless phases) as shown in sections 3.2 and 5. This then implies that pseudo-duality is not correct for all models of the type of equation (6.3) in all regions of the phase diagram and the replica argument indicates that this is the reason for the deviations from exact results for the quenched bond-diluted models.

The rest of this section will be devoted to an over-view of the results obtained so as to provide some perspective with regard to other results in this field.

(i) For pure, self-dual models, pseudo-duality on the square lattice is, in general, a necessary but not sufficient condition for (Kramers–Wannier) duality. For such models with at most two independent Boltzmann factors, pseudo-duality is equivalent to duality, however. In this case, pseudo-duality gives no new results.

(ii) For pure models, which are not self-dual, pseudo-duality gives new information on the phase diagrams of such models for the square lattice. These results are partly (i.e. not for all possible models) corroborated by high-temperature expansion results [4].

(iii) For pure models (self-dual or not) on the hexagonal and triangular lattices, pseudo-duality involves a star-triangle transformation. The results obtained are new, except for the Potts model, where they correspond to the exact ones.

(iv) All pure models treated here have symmetric Boltzmann factor matrices. Such a restriction to non-chiral models is not essential, however. For chiral models, see, e.g., the helical Potts model as treated by Kardar [26] or the recent work on the six-state chiral Potts model reported in [27].

(v) The pseudo-duality results always reduce to exact results in cases where the model reduces to the Ising or Potts models or to a decoupled set of such models. Further checks can be provided by an argument due to Kardar [26] and by the requirement of compatibility with the inversion relations known to hold for the Potts model on the chequerboard and (anisotropic) triangular lattices [28–32]. Preliminary calculations seem to indicate that such compatibilities are indeed respected.

(vi) For quenched bond-diluted systems, all results not pertaining to the square lattice are new. For the square lattice, the Ising and Potts model results are identical to the ones proposed by Nishimori [20] and Southern [19]. Since these are not exact (see section 4, the discussion above and also the review [33]), the results for the triangular and hexagonal lattices are also at most useful approximations. For this reason, no results for other diluted spin systems have been given.

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